

A GENERALIZATION OF A THEOREM OF FOXBY

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ABSTRACT. In this paper, it is proved that a commutative noetherian local ring admitting a finitely generated module of finite projective and injective dimensions with respect to a semidualizing module is Gorenstein. This result recovers a celebrated theorem of Foxby.

1. INTRODUCTION

A semidualizing module over a commutative noetherian ring is a common generalization of a free module of rank one and a dualizing module. Pioneering studies of semidualizing modules were done by Foxby [4], Golod [8] and Christensen [2]. A semidualizing module not only gives rise to several homological dimensions of modules, but also establishes an equivalence between categories of modules called Auslander and Bass categories. So far a lot of authors have studied semidualizing modules from various points of view.

Let R be a commutative noetherian ring, and let C be a semidualizing R -module. For a nonzero R -module M , the C -projective dimension $C\text{-pd}_R M$ of M is defined to be the infimum of integers n such that there exists an exact sequence

$$0 \rightarrow C \otimes_R P_n \rightarrow \cdots \rightarrow C \otimes_R P_1 \rightarrow C \otimes_R P_0 \rightarrow M \rightarrow 0$$

of R -modules where P_i is projective for $0 \leq i \leq n$. Dually, the C -injective dimension $C\text{-id}_R M$ of M is defined to be the infimum of integers n such that there exists an exact sequence

$$0 \rightarrow M \rightarrow \text{Hom}_R(C, I^0) \rightarrow \text{Hom}_R(C, I^1) \rightarrow \cdots \rightarrow \text{Hom}_R(C, I^n) \rightarrow 0$$

of R -modules where I^i is injective for $0 \leq i \leq n$. The C -projective and C -injective dimensions of the zero module are defined as $-\infty$.

In the 1970s, Foxby [5], verifying a conjecture of Vasconcelos [12], proved that a commutative noetherian local ring is Gorenstein if it admits a nonzero finitely generated module of finite projective and injective dimensions. As a natural generalization of this statement, Takahashi and White [11] asked whether the same conclusion holds true even if “projective” and “injective” are replaced with “ C -projective” and “ C -injective” respectively. Recently Sather-Wagstaff and Yassemi [10] answered that this question has an affirmative answer in the case where the C -projective dimension is equal to zero. The main purpose of this paper is to give a complete answer to the question; we shall prove the following theorem.

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Theorem 1.1. *Let R be a commutative noetherian ring, and let C be a semidualizing R -module. Let M be a finitely generated R -module with $C\text{-pd}_R M < \infty$ and $C\text{-id}_R M < \infty$. Then $R_{\mathfrak{p}}$ is Gorenstein for every $\mathfrak{p} \in \text{Supp}_R M$.*

2. PROOF OF THEOREM 1.1

We denote by $\mathcal{D}(R)$ the derived category of R . To prove our theorem, we give a lemma.

Lemma 2.1. *Let R be a commutative noetherian ring. Let X, Y, Z be R -complexes. Assume the following:*

- (1) $H_i(X)$ and $H_i(Z)$ are finitely generated for all $i \in \mathbb{Z}$,
- (2) $H_i(X)$ and $H_i(Z)$ are zero for all $i \ll 0$,
- (3) $\text{pd}_R Z < \infty$.

Then there is a natural isomorphism

$$\mathbf{R}\text{Hom}_R(X, Y) \otimes_R^{\mathbf{L}} Z \cong \mathbf{R}\text{Hom}_R(X, Y \otimes_R^{\mathbf{L}} Z).$$

in $\mathcal{D}(R)$.

Proof. There exist R -complexes

$$\begin{aligned} P &= (\cdots \rightarrow P_{a+1} \rightarrow P_a \rightarrow 0), \\ Q &= (0 \rightarrow Q_b \rightarrow Q_{b-1} \rightarrow \cdots \rightarrow Q_{c+1} \rightarrow Q_c \rightarrow 0) \end{aligned}$$

isomorphic (in $\mathcal{D}(R)$) to X and Z respectively, such that P_i and Q_j are finitely generated projective R -modules for $i \geq a$ and $b \geq j \geq c$. We have $\mathbf{R}\text{Hom}_R(X, Y) \otimes_R^{\mathbf{L}} Z = \text{Hom}_R(P, Y) \otimes_R Q$ and $\mathbf{R}\text{Hom}_R(X, Y \otimes_R^{\mathbf{L}} Z) = \text{Hom}_R(P, Y \otimes_R Q)$. There is a natural homomorphism $\text{Hom}_R(P, Y) \otimes_R Q \rightarrow \text{Hom}_R(P, Y \otimes_R Q)$ of R -complexes; see [1, (A.2.10)]. This homomorphism is an isomorphism by [3, (2.7)]. \square

Now we can prove our main theorem.

Proof of Theorem 1.1. Replacing R with $R_{\mathfrak{p}}$, we may assume that R is local. We denote by k the residue field of R . Note from [11, (2.9)–(2.11)] that M is in both the Auslander class $\mathcal{A}_C(R)$ and the Bass class $\mathcal{B}_C(R)$, and that $\text{Hom}_R(C, M)$ (respectively, $C \otimes_R M$) is a nonzero finitely generated R -module of finite projective (respectively, injective) dimension. We have isomorphisms

$$\begin{aligned} C \otimes_R M &\cong C \otimes_R^{\mathbf{L}} M \\ &\cong C \otimes_R^{\mathbf{L}} (C \otimes_R^{\mathbf{L}} \text{Hom}_R(C, M)) \\ &\cong (C \otimes_R^{\mathbf{L}} C) \otimes_R^{\mathbf{L}} \text{Hom}_R(C, M) \end{aligned}$$

in $\mathcal{D}(R)$. Using Lemma 2.1, we get isomorphisms

$$\begin{aligned} \mathbf{R}\text{Hom}_R(k, C \otimes_R M) &\cong \mathbf{R}\text{Hom}_R(k, (C \otimes_R^{\mathbf{L}} C) \otimes_R^{\mathbf{L}} \text{Hom}_R(C, M)) \\ &\cong \mathbf{R}\text{Hom}_R(k, C \otimes_R^{\mathbf{L}} C) \otimes_R^{\mathbf{L}} \text{Hom}_R(C, M). \end{aligned}$$

By [1, (A.7.9)], we obtain:

$$\begin{aligned} \sup(\mathbf{R}\text{Hom}_R(k, C \otimes_R^{\mathbf{L}} C)) &= \sup(\mathbf{R}\text{Hom}_R(k, C \otimes_R M)) - \sup(k \otimes_R^{\mathbf{L}} \text{Hom}_R(C, M)) \\ &= -\text{depth}_R(C \otimes_R M) - \text{pd}_R(\text{Hom}_R(C, M)) \in \mathbb{Z}, \\ \inf(\mathbf{R}\text{Hom}_R(k, C \otimes_R^{\mathbf{L}} C)) &= \inf(\mathbf{R}\text{Hom}_R(k, C \otimes_R M)) - \inf(k \otimes_R^{\mathbf{L}} \text{Hom}_R(C, M)) \\ &= -\text{id}_R(C \otimes_R M) \in \mathbb{Z}. \end{aligned}$$

Hence the R -complex $\mathbf{R}\mathrm{Hom}_R(k, C \otimes_R^{\mathbf{L}} C)$ is bounded, and so is $C \otimes_R^{\mathbf{L}} C$ by [6, (2.5)]. Thus we get $\mathrm{id}_R(C \otimes_R^{\mathbf{L}} C) = -\inf(\mathbf{R}\mathrm{Hom}_R(k, C \otimes_R^{\mathbf{L}} C)) \in \mathbb{Z}$ by [1, (A.5.7.4)]. It follows from [2, (4.4) and (4.6)(a)] that there is a natural isomorphism $C \cong \mathbf{R}\mathrm{Hom}_R(C, C \otimes_R^{\mathbf{L}} C)$, and so we have natural isomorphisms $\mathbf{R}\mathrm{Hom}_R(C \otimes_R^{\mathbf{L}} C, C \otimes_R^{\mathbf{L}} C) \cong \mathbf{R}\mathrm{Hom}_R(C, \mathbf{R}\mathrm{Hom}_R(C, C \otimes_R^{\mathbf{L}} C)) \cong \mathbf{R}\mathrm{Hom}_R(C, C) \cong R$. Therefore $C \otimes_R^{\mathbf{L}} C$ is a (semi)dualizing R -complex. It follows from [7, (3.2)] that C is isomorphic to R . Thus the dualizing R -complex $C \otimes_R^{\mathbf{L}} C$ is isomorphic to R , which concludes that R is a Gorenstein ring. \square

3. FOXBY'S THEOREM

Applying Theorem 1.1 to the semidualizing R -module $C = R$ for a local ring R , we immediately recover a well-known result of Foxby.

Corollary 3.1. [5, (4.4)] *Let R be a commutative noetherian local ring. Let M be a nonzero finitely generated R -module with $\mathrm{pd}_R M < \infty$ and $\mathrm{id}_R M < \infty$. Then R is Gorenstein.*

Remark 3.2. (1) Holm [9] investigates existence of modules of finite Gorenstein projective and injective dimensions, and shows a result which also recovers Corollary 3.1.

(2) Our method in the proof of Theorem 1.1 actually gives a more simple proof of Corollary 3.1 than the proof due to Foxby. In fact, let R and M be as in Corollary 3.1. Then we have

$$\mathbf{R}\mathrm{Hom}_R(k, M) \cong \mathbf{R}\mathrm{Hom}_R(k, R \otimes_R^{\mathbf{L}} M) \cong \mathbf{R}\mathrm{Hom}_R(k, R) \otimes_R^{\mathbf{L}} M,$$

which gives

$$\begin{aligned} \mathrm{id}_R R &= -\inf \mathbf{R}\mathrm{Hom}_R(k, R) \\ &= -\inf \mathbf{R}\mathrm{Hom}_R(k, M) + \inf(k \otimes_R^{\mathbf{L}} M) = \mathrm{id}_R M < \infty, \end{aligned}$$

namely, R is Gorenstein.

(3) Let R be a Cohen-Macaulay local ring with dualizing module ω . Then, applying Theorem 1.1 to the semidualizing R -module $C = \omega$, we get the same result as Corollary 3.1, because for an R -module M one has $\omega\text{-}\mathrm{pd}_R M < \infty$ (respectively, $\omega\text{-}\mathrm{id}_R M < \infty$) if and only if $\mathrm{id}_R M < \infty$ (respectively, $\mathrm{pd}_R M < \infty$); see [11, (2.11)].

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